How many degrees of freedom does Nature exploit?

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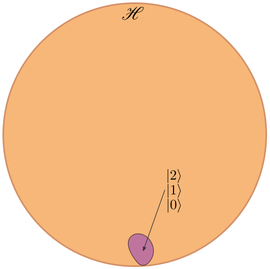

Summer school *The Nature of Entropy*
Saig, Germany
July 27th, 2019

Alexander von Humboldt
Stiftung/Foundation

MPQ

Starting point

Quantum mechanics is not just **weird** and **difficult**, it is also **powerful**. Why? and is it related to the fact that quantum mechanics exploits more degrees of freedom than classical mechanics?

Physical Church Turing Thesis

Weak physical Church Turing Thesis

Everything that can be computed by a physical machine can be computed by a Turing machine.

Strong physical Church Turing Thesis

Everything that can be **efficiently** computed by a physical machine can be **efficiently** computed by a Turing machine.

Physical Church Turing Thesis

Weak physical Church Turing Thesis

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Strong physical Church Turing Thesis

Everything that can be **efficiently** computed by a physical machine can be **efficiently** computed by a Turing machine.

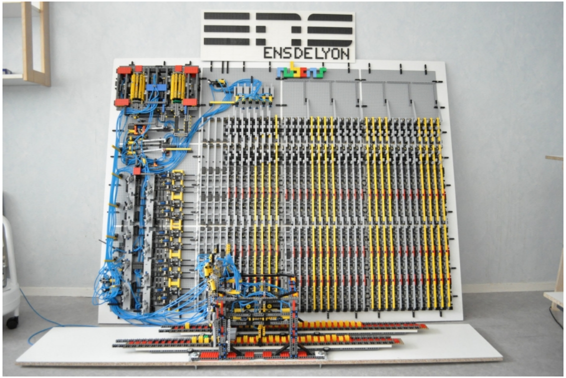
Example: factoring

$$\underbrace{19209192\cdots001}_{n \text{ digits}} = p \times q$$

Finding p and q can be done in time T

$$T \propto \exp\left(n^{1/3}\right)$$

Physical Church Turing Thesis

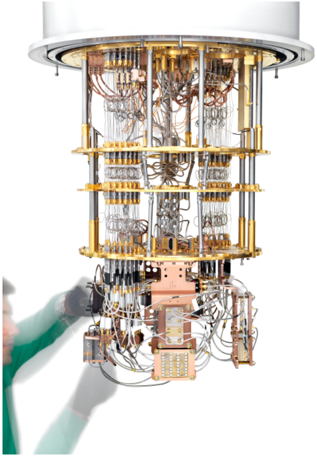


ENS Lyon – Lego Turing machine $\sim 10^{-2}$flops $\sim$ Oak ridge – Summit $\sim 10^{17}$ flops

$$t_{\text{Lego}} = C_{\text{Lego}} \exp\left(n^{1/3}\right)$$

$$t_{\text{Summit}} = C_{\text{Summit}} \exp\left(n^{1/3}\right)$$

But...



- Turing Machines with best algorithm

$$t = C \exp\left(n^{1/3}\right)$$

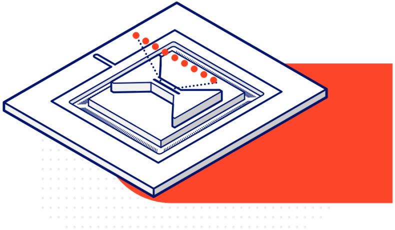
- Shor's algorithm on quantum bits

$$t \propto n^3$$

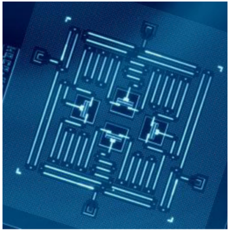
Quantum Turing Machines are believed to break the strongest form of the Church-Turing Thesis

Quantum advantage soon

Trapped ions
(ionQ, Maryland)



Superconducting circuits
(IBM, Rigetti, Google)



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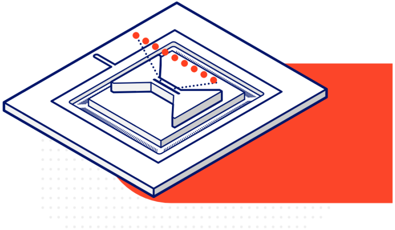

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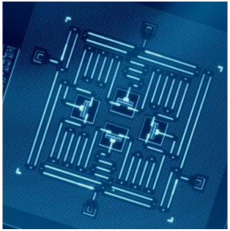

- in private, most near 50 qubits and $< 1\%$ error per gate
- on the cloud, *IBM Tokyo* 20 qubits, *Rigetti Aspen* 16 qubits

Where is quantum power coming from?

A few naive ideas for why quantum computers are stricly more powerful:

1. **Size** of the Hilbert space (exponentially bigger)
2. **Entanglement** between qubits
3. Coherence? Other?

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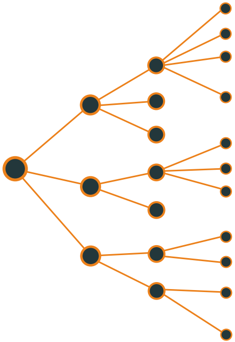
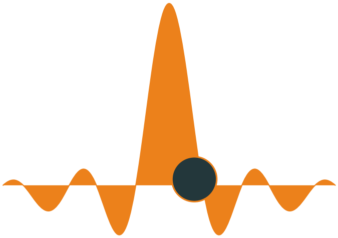
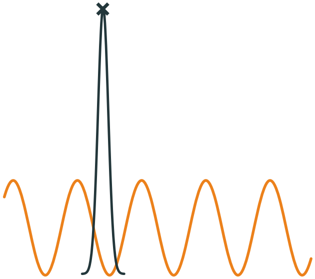
BUT

- Probability distributions $\mathbb{P}(i_1, i_2, \cdots, i_N)$ also have 2^N coefficients for N bits
- ψ is not just big, it has something dynamical to it

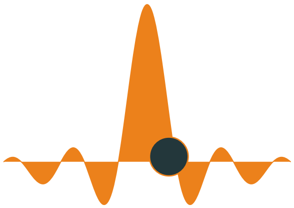
Understanding this power from quantum foundations?

Are there interpretations that give an easy understanding of the power of quantum mechanics?

In Bohmian mechanics

“Particles move” (but the laws are sorta weird)



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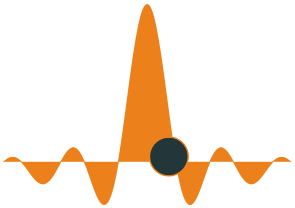

- Seems to be a mild modification of classical mechanics

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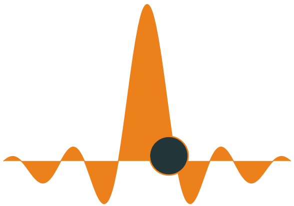

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- Still "mechanical", what else than a regular Turing machine could we build out of particles?

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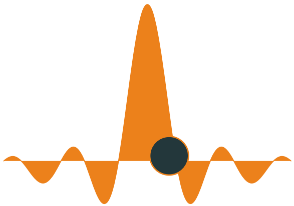
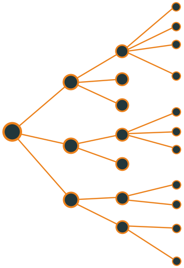


- Seems to be a mild modification of classical mechanics
- Still “mechanical”, what else than a regular Turing machine could we build out of particles?
- Naively leads to **underestimate** the power of quantum computing

In Many worlds

"The wave-function branches in measurement situations, in fact all the time, and the sorta branches are real worlds"



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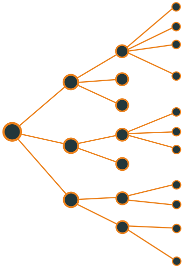
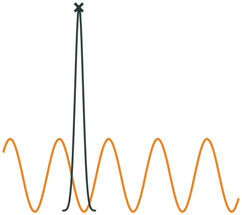


- DAAayyym soo many worlds to compute
- Intuitively gives massive parallelism that should allow huge speedups for NP-hard problems
- Naively **overestimates** the power of quantum computing (only $N \to \sqrt{N}$ for bruteforce algorithms like Grover search)

In collapse models

"Wave-functions collapse, like for real, put rarely for small stuff, and often for big stuff"



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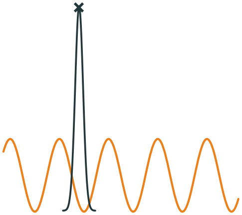
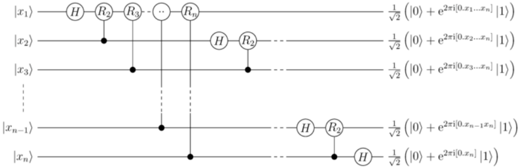


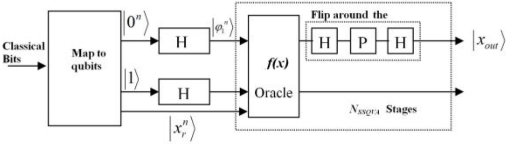
- Seems to strongly constrain the computational power of quantum mechanics
- Power of quantum computers should plateau quickly
- But no (because of fault tolerance)
- Naively **underestimates** the power of quantum computing

Only 3 peculiar building blocks in all algorithms

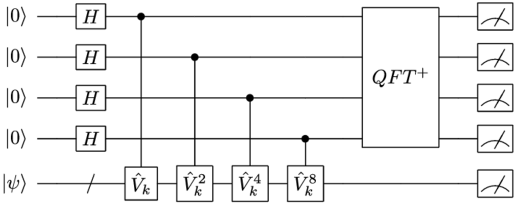
Quantum Fourier Transform



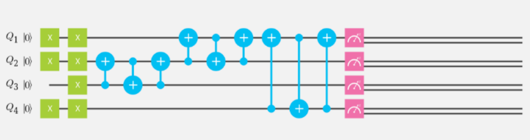
Amplitude amplification



Phase estimation

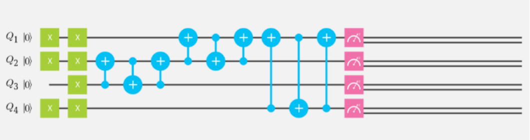


More subtleties



1. Only an **infinitely small** fraction of the Hilbert space is reachable by a sub-exponential set of gates
2. Clifford circuits construct non trivial massively entangled states **but** they are not stronger than classical computing

More subtleties



1. Only an **infinitely small** fraction of the Hilbert space is reachable by a sub-exponential set of gates
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Summary

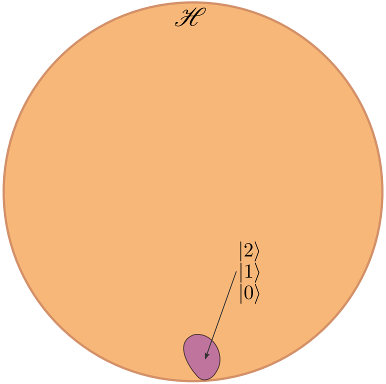
Understanding why quantum mechanics is powerful is a hard problem, it's not clear what resources are used

- Not just a size argument
- Not just massive parallelism
- Not simply entanglement

Question

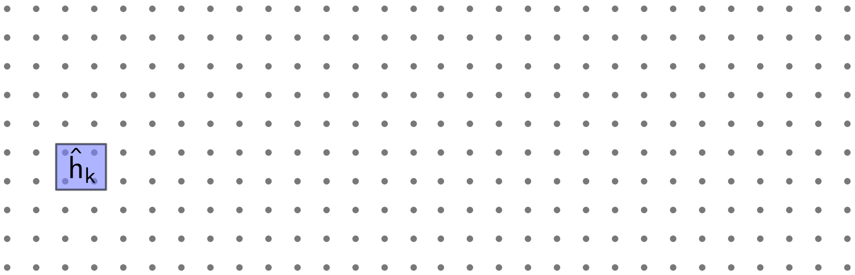
How much of the Hilbert space is used by Nature in

1. standard matter (many-body problem)
2. matter tricked into thinking (measurement based quantum computing)



$\simeq$ How many degrees of freedom does Nature exploit?

Many-body problem



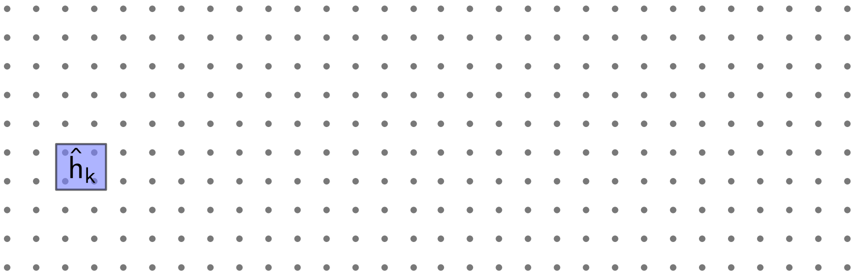
Problem

Finding low energy states of

$$\hat{H} = \sum_{k=1}^{N} \hat{h}_k$$

is **hard** because $\dim \mathscr{H} \propto D^N$

Many-body problem



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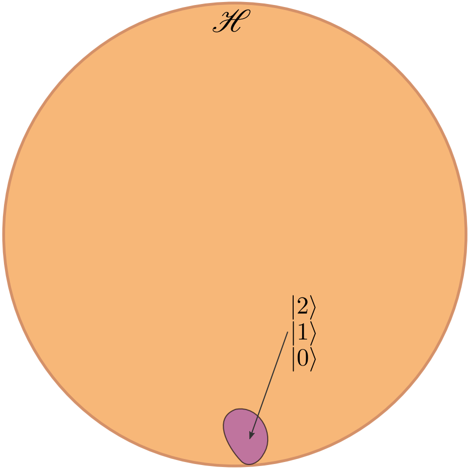
Possible solutions

- Perturbation theory
- Monte Carlo
- Bootstrap IR fixed point
- **Variational optimization** (*e.g.* Mean Field, TCSA, tensor networks)

Variational optimization



Generic (spin $D/2$) state $\in \mathscr{H}$:

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_N} |i_1, \cdots, i_N\rangle$$

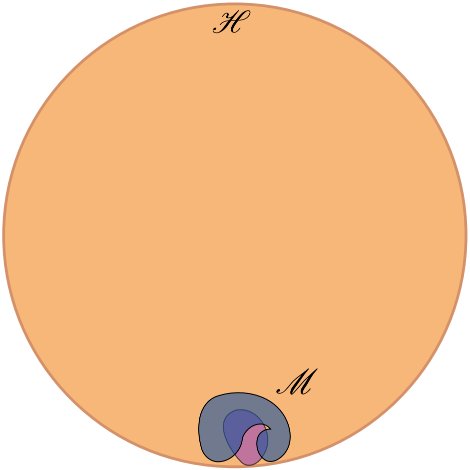
Exact variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{H}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

- $\dim \mathscr{H} = D^N$

Variational optimization



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Approx. variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{M}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

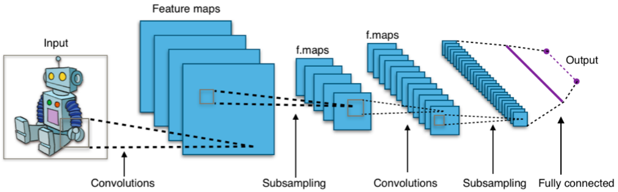
- $\dim \mathscr{M} \propto \mathrm{Poly}(N)$ or fixed

An idea popular in many fields

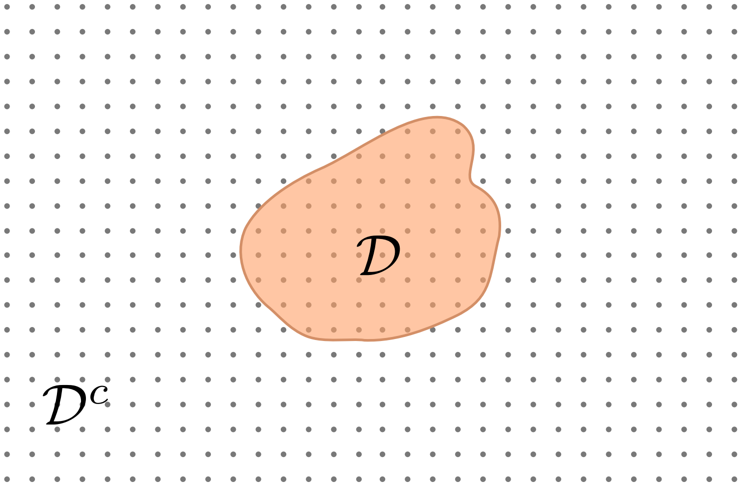
- **Mean field** approximation (of which TNS are an extension)

$$\psi(x_1, x_2, \cdots, x_n) = \psi_1(x_1)\,\psi_2(x_2)\cdots\psi_n(x_n)$$

- Special variational wave functions in **Quantum chemistry** (whole industry of ansatz)
- Fully connected and convolutional **neural networks** used in machine learning



Interesting states are weakly entangled



Low energy state

$|\psi\rangle = |0\rangle$ or $|1\rangle$ ...

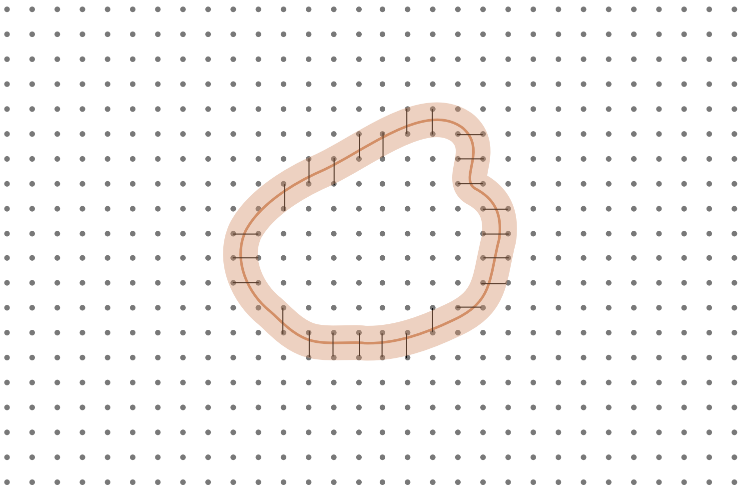
Reduced density matrix
$\rho = \mathrm{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

Entanglement entropy
$S = -\mathrm{tr}\left[\rho \log \rho\right]$

Area law

$$S \propto |\partial\mathcal{D}|$$

Interesting states are weakly entangled



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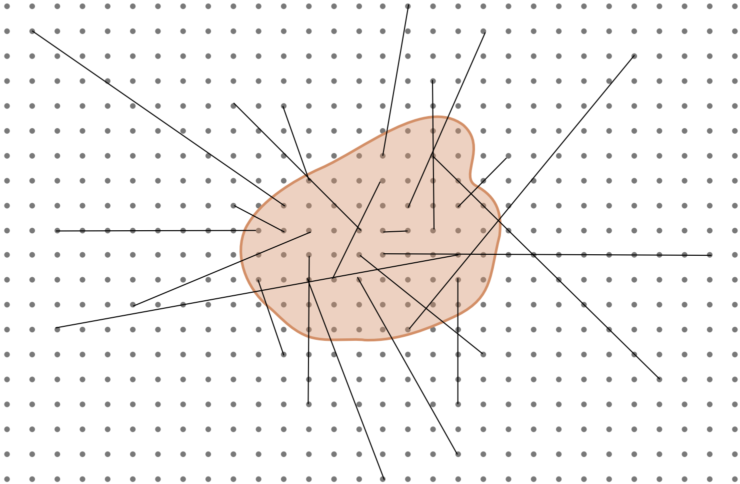
Typical states are strongly entangled



Random state

$|\psi\rangle = U_{\mathrm{Haar}}|\mathrm{trivial}\rangle$

Reduced density matrix

$\rho = \mathrm{tr}_{\mathcal{D}^c}\big[|\psi\rangle\langle\psi|\big]$

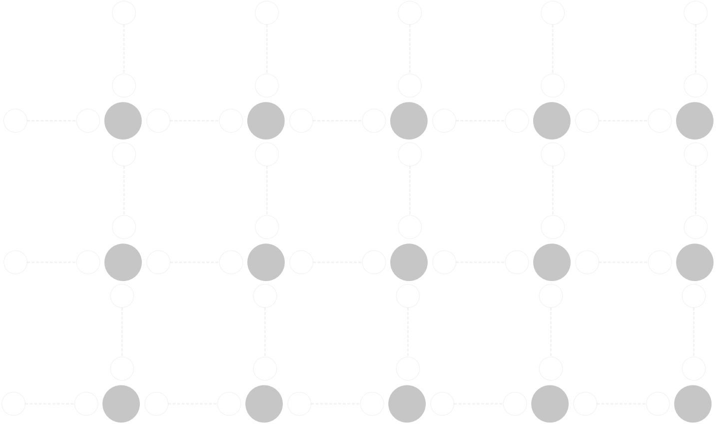
Entanglement entropy

$S = -\mathrm{tr}\big[\rho \log \rho\big]$

Volume law

$$S \propto |\mathcal{D}|$$

Constructing weakly entangled states



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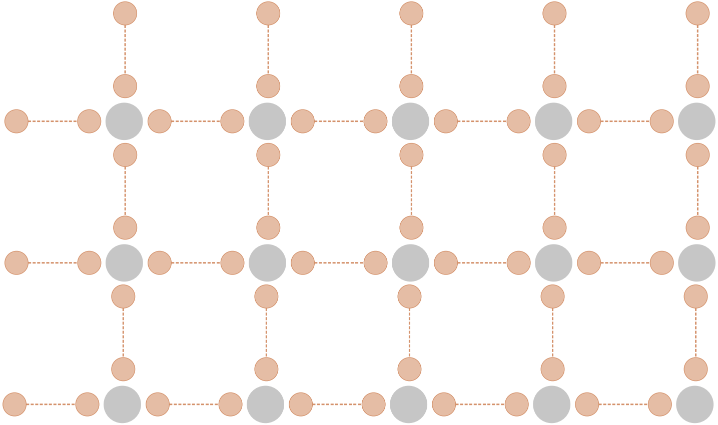
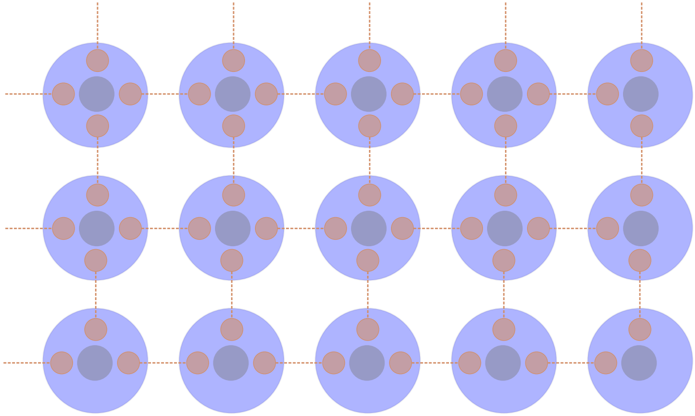


1. Put auxiliary **maximally entangled** states between sites

$$= \sum_{j=1}^{\chi} |j\rangle|j\rangle$$

Constructing weakly entangled states



1. Put auxiliary **maximally entangled** states between sites

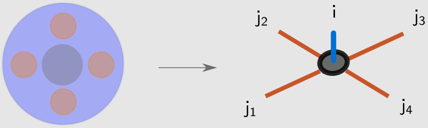
 $= \sum_{j=1}^{\chi} |j\rangle|j\rangle$

2. Map to initial Hilbert space on each site

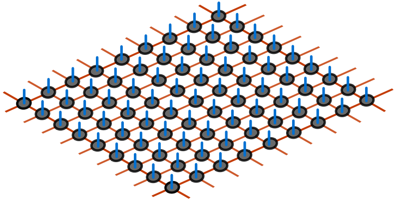
 $= A : \mathbb{C}^{4\chi} \to \mathbb{C}^D$

Tensor network states: definition

Why "tensor" network?



$$A : \mathbb{C}^{4\chi} \to \mathbb{C}^{d} \qquad \longrightarrow \qquad A^{i}_{j_1,j_2,j_3,j_4}$$

$|A\rangle =$ 

with tensor contractions on links

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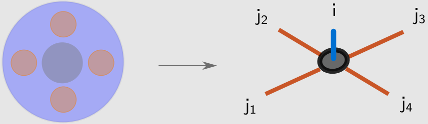

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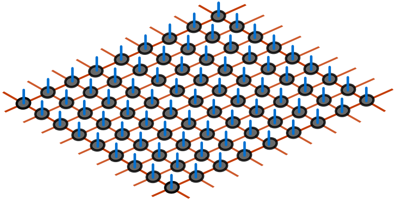
$|A\rangle =$ 

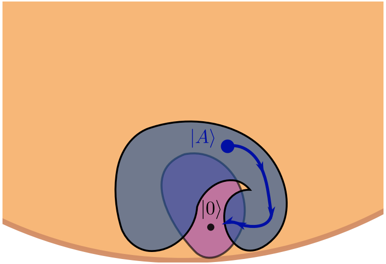
with tensor contractions on links

Optimization

Find best A for fixed χ $\quad$ ($D \times \chi^4$ coeff.)

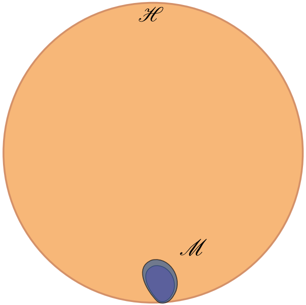
$$E_0 \simeq \min_{A} \frac{\langle A|\widehat{H}|A\rangle}{\langle A|A\rangle}$$

for example go down $\dfrac{\partial E}{\partial A^{i}_{j_1,j_2,j_3,j_4}}$



Some facts

$d = 1$ spatial dimension



Theorems (colloquially)

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ with χ fixed
2. **All** $|A\rangle$ are ground states of gapped H

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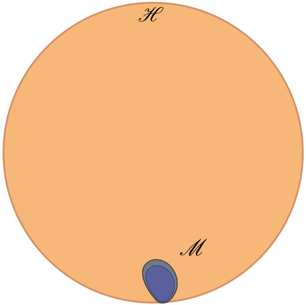
$d = 1$ spatial dimension



Theorems (colloquially)

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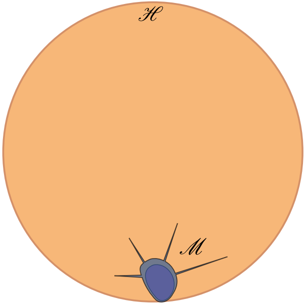
$d \geqslant 2$ spatial dimension



Folklore

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ with χ fixed
2. **Most** $|A\rangle$ are ground states of gapped H

Uses and limitations

Uses today

- Understanding QCD (via Hamiltonian lattice gauge theory)
- Understanding toy models of High T_c superconductivity

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- Understanding QCD (via Hamiltonian lattice gauge theory)
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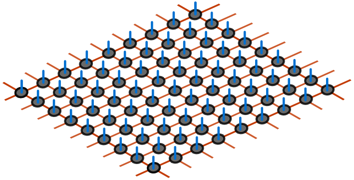
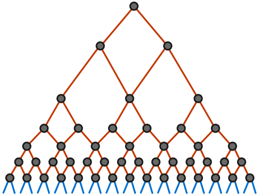
Why it doesn't solve everything

In $d \geqslant 2$ one can have:

- $|A\rangle$ known
- *e.g.* $\langle A|\hat{\mathcal{O}}_i\hat{\mathcal{O}}_j|A\rangle$ impossible to compute exactly in general
- yet uncontroled approximations seem to work with (arbitrary?) precision for physical systems

Tensor network states

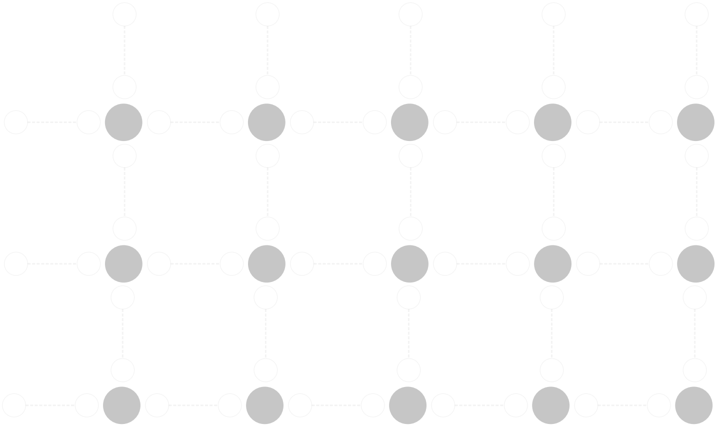




Summary

Compact parameterization of quantum states (few degrees of freedom) which can approximate well low energy states of quantum systems with local interactions

Measurement based quantum computing



1. Take N qubits in a quantum state $|\psi\rangle$

Measurement based quantum computing

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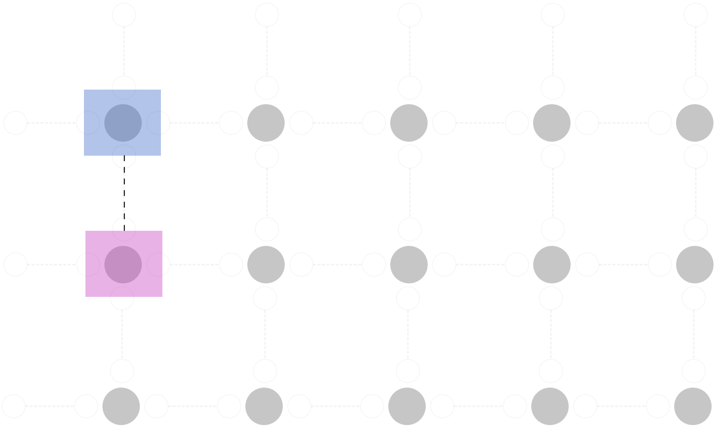

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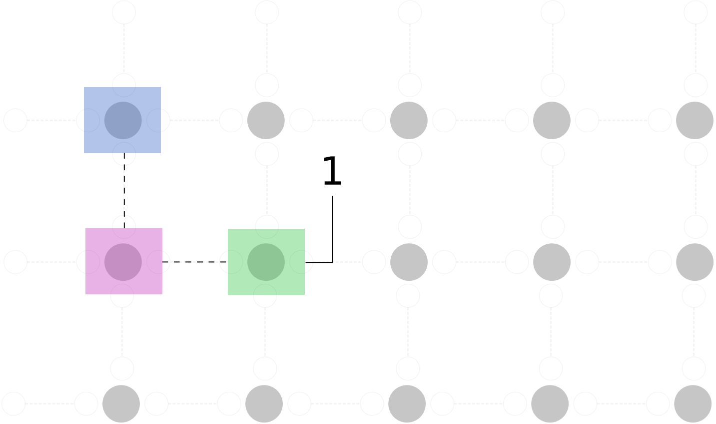
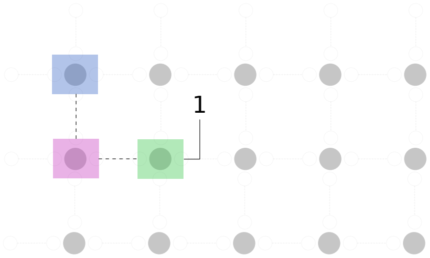
3. Measure a second qubit with a measurement depending on r

Measurement based quantum computing



1. Take N qubits in a quantum state $|\psi\rangle$
2. Measure one qubit, get a result r
3. Measure a second qubit with a measurement depending on r
4. Keep on for a while and get a final measurement result

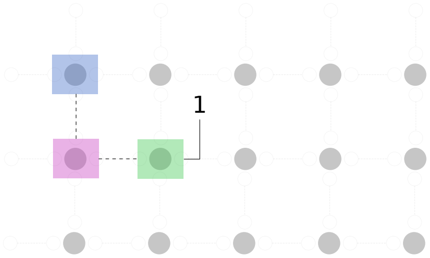
States needed for universal quantum computation



Problem

What is the computational power of "measurement based quantum computing" depending on the initial state $|\psi\rangle$?

States needed for universal quantum computation



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- If ψ is a product (no entanglement) $\chi = 1$, no power

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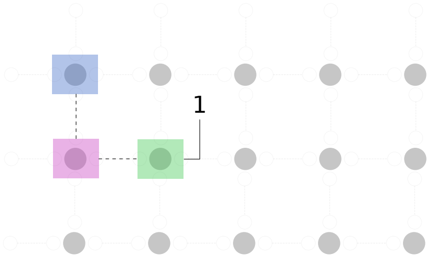

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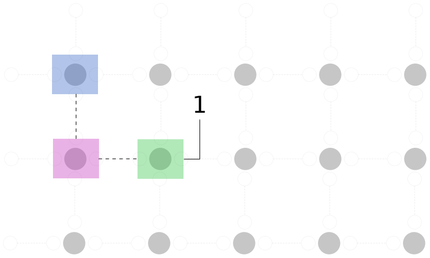
States needed for universal quantum computation



Problem

What is the computational power of “measurement based quantum computing” depending on the initial state $|\psi\rangle$?

- If ψ is a product (no entanglement) $\chi = 1$, no power
- If ψ is the cluster state, i.e. a tensor network with only $\chi = 2$, MBQC has the power of a general quantum Turing machine
- Some ψ that are far more entangled have no power

Conclusion

In many instances, in intert and computing matter, Nature does seem to use very little of the Hilbert space (tensor network states with small χ).

As a result, quantum mechanics is just a bit more difficult, and just a bit more powerful than classical mechanics. This "just a bit" is not well understood.

Example of questions

- What quantum states are universal for measurement based quantum computing?
- Can all the translation invariant quantum systems we see in Nature ultimately be efficiently classically simulated?
- Is there a formulation of quantum mechanics that makes its computational power clearer from the start?